

THE CLASSIFYING SPACE OF AN INVERSE SEMIGROUP

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ABSTRACT. We refine Funk's description of the classifying space of an inverse semigroup by replacing his $*$ -semigroups by right generalized inverse $*$ -semigroups. Our proof uses the idea that presheaves of sets over meet semi-lattices may be characterized algebraically as right normal bands.

1. STATEMENT OF THE THEOREM

With each inverse semigroup S , we shall associate two categories, the aim of this paper being to prove that these two categories are equivalent.

To define the first, we need the concept of an étale action of an inverse semigroup. These were first explicitly defined in [4], but their origins lie in [8, 11] and they played an important role in [10]. Let X be a non-empty set. A *left S -action* of S on X is a function $S \times X \rightarrow X$, defined by $(s, x) \mapsto s \cdot x$ (or sx), such that $(st)x = s(tx)$ for all $s, t \in S$ and $x \in X$. If S acts on X we say that X is an *S -set*. In this paper, all actions will be assumed left actions. A *left étale action* (S, X, p) of S on X is defined as follows [4, 12]. Let $E(S)$ denote the semilattice of idempotents of S . There is a function $p: X \rightarrow E(S)$ and a left action $S \times X \rightarrow X$ such that the following two conditions hold:

- (E1) $p(x) \cdot x = x$;
- (E2) $p(s \cdot x) = sp(x)s^{-1}$.

The set X is also partially ordered when we define $x \leq y$ when there exists $e \in E(S)$ such that $x = e \cdot y$. A *morphism* $\varphi: (S, X, p) \rightarrow (S, Y, q)$ of left étale actions is a map $\varphi: X \rightarrow Y$ such that $q(\varphi(x)) = p(x)$ for any $x \in X$ and $\varphi(s \cdot x) = s \cdot \varphi(x)$ for any $s \in S$ and $x \in X$. The category of all left étale S -actions is called the *classifying space* or *classifying topos* of S and is denoted by $\mathcal{B}(S)$. This space is the subject of Funk's paper [3].

In the last section, we shall need a more general notion of morphism. Let (S, X, p) and (T, Y, q) be étale actions where we do not assume that S and T are the same. Then $(\alpha, \beta): (S, X, p) \rightarrow (T, Y, q)$ is called a *morphism* if $\alpha: S \rightarrow T$ is a semigroup homomorphism, $\beta: X \rightarrow Y$ is a function such that $q(\beta(x)) = p(x)$, and $\beta(s \cdot x) = \alpha(s) \cdot \beta(x)$.

To define our second category, we need some definitions from semigroup theory. An element s of a semigroup S is said to be (*von Neumann*) *regular* if there is an element t , called an *inverse* of s , such that $s = sts$ and $t = tst$. The set of inverses of the element s is denoted by $V(s)$. In an inverse semigroup S , we write the unique inverse of s as s^{-1} and we define $\mathbf{d}(s) = s^{-1}s$ and $\mathbf{r}(s) = ss^{-1}$. A *band* is a semigroup in which every element is idempotent and a *right normal band* is a band satisfying the identity $efg = feg$. A *right generalized inverse semigroup* is a regular semigroup whose set of idempotents is a right normal band. On a regular semigroup S , we may define a relation $a \leq b$ if and only if $a = eb = bf$ for some idempotents e and f . This is a partial order called the *natural partial order*. The order need not be compatible with the multiplication but it is precisely

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when the semigroup S is *locally inverse* meaning that each local submonoid, eSe , where e is an idempotent, is inverse. If S is a band, the order is the usual order on idempotents: $e \leq f$ if and only if $e = ef = fe$. For right generalized inverse semigroups in general, and right normal bands in particular, this partial order is compatible with the multiplication.

The following definition is taken from [3], except for (S4) which is new. A semigroup S is said to be a *right $*$ -semigroup* if it is equipped with a unary operation $s \mapsto s^*$ satisfying the following axioms:

- (S1) $(s^*)^* = s$.
- (S2) $s^* \in V(s)$.
- (S3) $(st)^* = t^*(stt^*)^*$.
- (S4) If $e^2 = e$ then $e^* = e$.

Left $*$ -semigroups can be defined in an analogous way. In this paper, we shall only be interested in right $*$ -semigroups and so we shall omit the word *right* in what follows. Clearly, these semigroups are regular. Inverse semigroups are special examples where the $*$ is just inversion. We shall be interested in *right generalized inverse $*$ -semigroups*. *Homomorphisms* of $*$ -semigroups are defined in the obvious way and will sometimes be called *$*$ -homomorphisms*. A semigroup homomorphism from a $*$ -semigroup to an inverse semigroup automatically preserves the $*$ operation.

It is worth mentioning that axioms (S2) and (S3) arise in a completely different context in the work of Tom Blyth [1, 2].

Let T be a right generalized inverse $*$ -semigroup and S an inverse semigroup. A semigroup homomorphism $\theta: T \rightarrow S$ is said to be *étale* [3] if for each $e \in E(T)$ the restriction map $(\theta \mid Te): Te \rightarrow S\theta(e)$ is a bijection. We denote by Et/S the category of right generalized inverse $*$ -semigroups étale over S ; the objects of this category are étale homomorphisms $\phi: T \rightarrow S$, and a morphism from ϕ_1 to ϕ_2 is a homomorphism $\theta: T_1 \rightarrow T_2$ satisfying $\phi_1\theta = \phi_2$. The theorem we shall prove is the following.

Theorem 1.1. *Let S be an inverse semigroup. Then the classifying space of S is equivalent to the category of right generalized inverse $*$ -semigroups étale over S .*

The key idea lying behind the work of this paper can be traced back to Wagner [13] and it is that presheaves of sets over meet semilattices can be regarded as right normal bands. This idea is explored in more detail in [7]. For results on general semigroup theory see [6] and for inverse semigroups [9].

2. PROOF OF THE THEOREM

A regular semigroup is *orthodox* if its set of idempotents forms a band. On an orthodox semigroup S , the relation γ defined by

$$s \gamma t \Leftrightarrow V(s) \cap V(t) \neq \emptyset \Leftrightarrow V(s) = V(t)$$

is the minimum inverse congruence. As usual, we denote Green's relations on any semigroup by $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and \mathcal{J} . The \mathcal{L} -class containing the element a is traditionally denoted L_a . Right generalized inverse semigroups have a right normal band of idempotents. We may deduce from this that in such a semigroup $efa = fea$ for any idempotents e and f and any element a .

An important property of right generalized inverse semigroups is described below. It is the beginning of the process of characterizing étale maps.

Lemma 2.1. *Let S be a right generalized inverse semigroup.*

- (1) *If $a, b \in Se$, where e is an idempotent, and $\gamma(a) = \gamma(b)$ then $a = b$.*
- (2) *If $a \gamma a^2$ then $a = a^2$. Thus γ is idempotent pure.*

- (3) Let $\gamma(a)\gamma(e) = \gamma(a)$, where $\gamma(e)$ is an idempotent. Then there exists $b \in Se$ such that $\gamma(b) = \gamma(a)$.
- (4) The natural homomorphism $S \rightarrow S/\gamma$ is étale when S is a right generalized inverse $*$ -semigroup.
- (5) Let $*$ be a unary operation on S that satisfies (S1), (S2) and (S3). Then (S4) holds.

Proof. (1). Let $a' \in V(a)$. Then from $ae = a$ we get that $a'ae = a'a$. It follows that $ea'a \leq e$ and $ea'a \mathcal{L} a'a$. But in a right normal band the \mathcal{L} -relation is equality. It follows that $a'a = ea'a$ and so $a'a \leq e$. If $b' \in V(b)$, we may similarly deduce that $b'b \leq e$. Right generalized inverse semigroups are locally inverse and so $a'ab'b = b'ba'a$. We now use the fact that $V(a) = V(b)$. We therefore have

$$a'b = a'aa' \cdot b = a'a \cdot a'b = a'b \cdot a'a = a'ba' \cdot a = a'a.$$

It follows that

$$a = a \cdot a'a \cdot a'a = a \cdot a'b \cdot a'b = aa' \cdot ba' \cdot b = ba' \cdot aa' \cdot b = ba' \cdot ba' \cdot b = b$$

as required.

(2). The elements a and a^2 have the same image under γ and $a, a^2 \in Sa'a$ where $a' \in V(a)$. It follows by (1) that $a = a^2$.

(3). By (2) above we know that e is an idempotent. It is therefore enough simply to define $b = ae$.

(4). This is immediate by (1) and (3) above and the definition of étale.

(5). From (S1), (S2) and the fact that γ is a homomorphism, it follows that $\gamma(s^*) = \gamma(s)^{-1}$ for any $s \in S$. Let $e \in E(S)$. Then $\gamma(e^*) = \gamma(e)$. It follows that e^* is an idempotent since γ is idempotent pure by (2) above. Since $e\gamma e^*$ and we are in a right normal band, we have that $e^*e = e$. Applying (S3), we obtain

$$e^* = (e^*e)^* = e^*(e^*ee^*)^* = e^*e = e.$$

It follows that (S4) holds. \square

We shall now describe the form taken by the natural partial order on a regular semigroup in the case important to us.

Lemma 2.2. *Let S be a right generalized inverse $*$ -semigroup.*

- (1) $a \leq b$ if and only if $a^* \leq b^*$.
- (2) $a \leq b$ if and only if $a = aa^*b$ if and only if $a^* = a^*ab^*$.

Proof. (1). Let $a \leq b$. By definition, we have that $a = eb = bf$ for some idempotents e and f . By (S3), we have that $a^* = (eb)^* = b^*(ebb^*)^*$. But ebb^* is an idempotent, and so by (S4) we have that $a^* = b^*(ebb^*) = b^*e'$ where e' is an idempotent. We shall now prove that $a^* = b^*bfb^*$. This follows from Lemma 2.1 using the fact that $a^* \in Sbb^*$ and $\gamma(b^*bfb^*) = \gamma(a)^{-1}$. The proof of the converse is immediate.

(2) Suppose that $a \leq b$. From $a \leq b$ we have that $a = eb = bf$ for some idempotents e and f . But then $ea = a$ and so $ea a^* = aa^*$. It follows that $a = aa^*eb = eaa^*b = aa^*b$ using the fact that the idempotents of S form a right normal band. Conversely, suppose that $a = aa^*b$. Then $\gamma(a) \leq \gamma(b)$ and so $\gamma(a) = \gamma(ba^*a)$. But $a, ba^*a \in Sa^*a$. It follows by Lemma 2.1 that $a = ba^*a$ and so $a \leq b$, as required. The proof of the other equivalence is now immediate by this result and (1). \square

It follows that the natural partial order on a right generalized inverse $*$ -semigroup coincides with the order studied in [3].

Let $\theta: S \rightarrow T$ be a surjective homomorphism of regular semigroups. We say that it is an \mathcal{L} -cover if for each idempotent $e \in S$ the map $(\theta \mid L_e): L_e \rightarrow L_{\theta(e)}$ is bijective. We could prove some of the results that follow in greater generality.

Proposition 2.3. *Let S be a right generalized inverse semigroup.*

- (1) *The natural map $S \rightarrow S/\gamma$ is an \mathcal{L} -cover.*
- (2) *There is a bijection between S and the subset of $S/\gamma \times E(S)$ consisting of those pairs $(\gamma(s), e)$ where $s's\gamma e$ and $s' \in V(s)$.*

Proof. (1) Suppose first that $s\mathcal{L}t$ and $\gamma(s) = \gamma(t)$. Let $s' \in V(s)$. Then $s\mathcal{L}s's$ and $t\mathcal{L}s't$ since $s' \in V(t)$. It follows that $s's\mathcal{L}s't$. But in a right normal band the \mathcal{L} -relation is just equality and so $s's = s't$. We have shown that $s, t \in Ss's$. Thus by Lemma 2.1, we have that $s = t$, as required.

Next, let $e \in E(S)$ and $\gamma(t)\mathcal{L}\gamma(e)$. Let $t' \in V(t)$. Then $\gamma(t't)\mathcal{L}\gamma(e)$. Since both elements are idempotent, we have that $t't\gamma e$. It follows that $e = et'te$ and $t't = t'tet't$. Consider the element $te \in S$. Then $\gamma(te) = \gamma(t)\gamma(e) = \gamma(t)$. From $e = (et')te$ it is immediate that $te\mathcal{L}e$.

(2) Put $S/\gamma * E(S)$ equal to the set of ordered pairs satisfying the condition. Define $\kappa: S \rightarrow S/\gamma * E(S)$ by $\kappa(s) = (\gamma(s), s's)$. This is well-defined since in a right normal band the \mathcal{L} -relation is equality. The fact that κ is a bijection follows by (1) above. \square

In our next result, we characterize étale homomorphisms.

Proposition 2.4. *Let S be an inverse semigroup and T a right generalized inverse $*$ -semigroup.*

- (1) *Let $\theta: T \rightarrow S$ be an étale homomorphism. Then the image of θ is a left ideal of S .*
- (2) *Let $\theta: T \rightarrow S$ be a homomorphism such that whenever $a, b \in Te$, where e is an idempotent, and $\theta(a) = \theta(b)$ then $a = b$. Then $\ker(\theta) = \gamma$.*
- (3) *Let $\theta: T \rightarrow S$ be a homomorphism whose kernel is γ and whose image is a left ideal of S . Then θ is étale.*

Proof. (1). Let $x \in \text{im}(\theta)$ and $s \in S$. We prove that $sx \in \text{im}(\theta)$. Let $x = \theta(t)$ where $t \in T$. Put $e = t^*t$. Then by assumption, $(\theta \mid Te): Te \rightarrow S\theta(e)$ is a bijection. But $x = x\theta(e)$ and so $sx \in S\theta(e)$. It follows that there is a $u \in T$ such that $\theta(u) = sx$, as required.

(2). Let $\gamma(a) = \gamma(b)$. Then $a^* \in V(b)$. Thus $b = ba^*b$ and $a^* = a^*ba^*$. It follows that $\theta(a)^{-1} = \theta(b)^{-1}$ and so $\theta(a) = \theta(b)$. Conversely, let $\theta(a) = \theta(b)$. Then $\theta(a^*) = \theta(a^*)\theta(b)\theta(a^*)$ and $\theta(b) = \theta(b)\theta(a^*)\theta(b)$. Thus by assumption, $a^* = a^*ba^*$ and $b = ba^*b$. We have shown that $V(a) \cap V(b) \neq \emptyset$ and so $\gamma(a) = \gamma(b)$.

(3). Let $e \in E(T)$. We need to prove that $(\theta \mid Te): Te \rightarrow S\theta(e)$ is a bijection. Let $x \in S\theta(e)$. Then $x = x\theta(e)$. But by assumption, the image of θ is a left ideal and so $x \in \text{im}(\theta)$. Thus there exists $u \in T$ such that $\theta(u) = x$. Now observe that $\theta(ue) = x$. It follows that $(\theta \mid Te)$ is surjective. Now let $t_1, t_2 \in Te$ such that $\theta(t_1) = \theta(t_2)$. We prove that $t_1 = t_2$. Observe that both $t_1^*t_1$ and $t_1^*t_2$ are idempotents less than or equal to e and so commute. But $\theta(t_1t_1^*) = \theta(t_2t_1^*)$ and so $t_1t_1^* = t_2t_1^*$ by Lemma 2.1(1). It follows that

$$t_1^*t_1 = t_1^* \cdot t_1t_1^* \cdot t_1 = t_1^* \cdot t_2t_1^* \cdot t_1 = t_1^*t_1 \cdot t_1^*t_2 = t_1^*t_2.$$

Therefore

$$t_1 = t_1t_1^*t_1 = t_1t_1^*t_2 = t_2t_1^*t_2 = t_2.$$

\square

We now begin the proof of our main theorem. Let (S, X, p) be a left étale action. Define the set

$$S * X = \{(s, x) \in S \times X : \mathbf{d}(s) = p(x)\}.$$

Proposition 2.5. *Let (S, X, p) be a left étale action. On the set $S * X$ define*

$$(s, x)(t, y) = (st, \mathbf{d}(st) \cdot y)$$

*and denote by $\pi_X: S * X \rightarrow S$ the projection map $(s, x) \mapsto s$. Define $(s, x)^* = (s^{-1}, s \cdot x)$.*

- (1) *$S * X$ is a right generalized inverse $*$ -semigroup whose idempotents are precisely the elements of the form $(p(x), x)$.*
- (2) *On the regular semigroup $S * X$ the natural partial order $(s, x) \leq (t, y)$ is given by $s \leq t$ and $x \leq y$.*
- (3) *The projection map π_X is étale. It is surjective if and only if the action has global support.*
- (4) *Let $\theta: X \rightarrow Y$ be a morphism of étale actions (S, X, p) and (S, Y, q) . Then $\bar{\theta}: S * X \rightarrow S * Y$ defined by $(s, x) \mapsto (s, \theta(x))$ is a homomorphism of $*$ -semigroups and $\pi_Y \bar{\theta} = \pi_X$.*
- (5) *We have constructed a functor from $\mathcal{B}(S)$ to Et/S .*

Proof. (1) The proof of associativity is pleasantly trivial. Observe that elements of the form $(p(x), x)$ are idempotents since

$$(p(x), x)(p(x), x) = (p(x), p(x) \cdot x) = (p(x), x).$$

Conversely, if (s, x) is an idempotent then we have that $s = s^2$ and $x = \mathbf{d}(s) \cdot x$. It follows that $s = e$ is an idempotent and that $e = p(x)$. We put $(s, x)^* = (s^{-1}, s \cdot x)$ and this is well-defined since $p(s \cdot x) = sp(x)s^{-1} = ss^{-1}ss^{-1} = (s^{-1})$. It is routine to check that our axioms for a $*$ -semigroup hold. A simple calculation shows that $(p(x), x)(p(y), y) = (p(x)p(y), p(x) \cdot y)$. It readily follows that the set of idempotents forms a right normal band. We have therefore shown that $S * X$ is a right generalized inverse $*$ -semigroup.

(2) Suppose that $(s, x) \leq (t, y)$. By Lemma 2.2, we have that

$$(s, x) = (t, y)(s, x)^*(s, x) \text{ and } (s, x)^*(s, x) = (s, x)^*(s, x)(t, y)^*(t, y).$$

This quickly reduces to $s \leq t$ and $x \leq y$. Conversely, suppose that $s \leq t$ and $x \leq y$. Then

$$(s, x)(s, x)^*(t, y) = (s, x)(s^{-1}, s \cdot x)(t, y) = (s, \mathbf{d}(s) \cdot y) = (s, x).$$

Applying Lemma 2.2, it follows that $(s, x) \leq (t, y)$.

(3) It is immediate that the projection map is a homomorphism and that its kernel is γ . We show that its image is a left ideal. Let $s = \pi_X(s, x)$ and $t \in S$. Then $\mathbf{d}(ts) \leq \mathbf{d}(s)$ and therefore $ts = \pi_X(ts, \mathbf{d}(ts) \cdot x)$. We now apply Proposition 2.4. The fact that the projection map is surjective if and only if the action has global support is easy to check.

(4) The map $\bar{\theta}$ is well-defined since $q(\theta(x)) = p(x)$. It is a homomorphism because $\theta(s \cdot x) = s \cdot \theta(x)$. The proofs of the remaining claims are straightforward.

(5) This is now routine. \square

We now construct a functor going in the opposite direction.

Proposition 2.6. *Let T be a right generalized inverse $*$ -semigroup, S an inverse semigroup and $\theta: T \rightarrow S$ an étale homomorphism.*

- (1) *Define $S \times E(T) \rightarrow E(T)$ by*

$$s \cdot e = tt^*$$

*where $t^*t \leq e$ and $\theta(t) = s\theta(e)$. Also define $p: E(T) \rightarrow E(S)$ by $p(e) = \theta(e)$. Then $(S, E(T), p)$ is a left étale action.*

- (2) *Let $\alpha: T_1 \rightarrow T_2$ be a $*$ -homomorphism of étale maps from $\theta_1: T_1 \rightarrow S$ to $\theta_2: T_2 \rightarrow S$. Then $\bar{\alpha} = (\alpha | E(T_1)): E(T_1) \rightarrow E(T_2)$ is a morphism of étale actions.*

(3) We have constructed a functor from Et/S to $\mathcal{B}(S)$.

Proof. (1) Because the map θ is étale the element t is uniquely defined. We prove first that $s \cdot (t \cdot e) = (st) \cdot e$. By definition, $(st) \cdot e = cc^*$ where $c^*c \leq e$ and $\theta(c) = st\theta(e)$. In addition, $t \cdot e = bb^*$ where $b^*b \leq e$ and $\theta(b) = t\theta(e)$, and $s \cdot bb^* = aa^*$ where $a^*a \leq bb^*$ and $\theta(a) = s\theta(bb^*)$. Observe that $(ab)^*ab = b^*(abb^*)^*ab = b^*a^*ab$ since $a^*a \leq bb^*$. But $b^*a^*ab \leq b^*b \leq e$. Thus $(ab)^*ab \leq e$. In addition, $\theta(ab) = \theta(a)\theta(b) = s\theta(bb^*)t\theta(e) = st\theta(e)$. It follows by uniqueness that $c = ab$. But $cc^* = ab(ab)^* = abb^*(abb^*)^* = aa^*$. Where throughout we have used axiom (S3) for $*$ -semigroups. We have an action, we now prove that it is an étale action.

(E1) holds: $p(e) \cdot e = aa^*$ where $a^*a \leq e$ and $\theta(a) = p(e)\theta(e) = p(e)$. It follows by uniqueness and axiom (S4) that $a = e$ and so $p(e) \cdot e = e$, as required.

(E2) holds: $p(s \cdot e) = \theta(s \cdot e)$. Let $t^*t \leq e$ and $\theta(t) = sp(e)$. Now $\theta(tt^*) = \theta(t)\theta(t)^{-1} = sp(e)s^{-1}$, as required.

(2) Since α is a homomorphism of semigroups $\bar{\alpha}$ is a well-defined map. We have to show that it is a map of étale actions. Let $s \cdot e = aa^*$ where $a^*a \leq e$, and $\theta_1(a) = s\theta_1(e)$. Let $s \cdot \bar{\alpha}(e) = bb^*$ where $b^*b \leq \bar{\alpha}(e)$, and $\theta_2(b) = s\theta_2(\bar{\alpha}(e)) = s\theta_1(e)$. But $\alpha(a)^*\alpha(a) \leq \bar{\alpha}(e)$ and $\theta_2(\alpha(a)) = \theta_2(b)$. It follows by uniqueness that $\alpha(a) = b$. Hence $\bar{\alpha}(s \cdot e) = s \cdot \bar{\alpha}(e)$. It now readily follows that $\bar{\alpha}$ is a morphism of left étale actions.

(3) The proof of this is routine. \square

It only remains to show that the two functors defined above determine an equivalence of categories.

Proposition 2.7.

- (1) Let (S, X, p) be a left étale action. Then this is isomorphic to the left étale action constructed from $\pi_X: S * X \rightarrow S$.
- (2) Let $\theta: T \rightarrow S$ be an étale homomorphism from a right generalized $*$ -semigroup to an inverse semigroup. Then this is isomorphic to the étale map constructed from the left étale action $(S, E(T), p)$.

Proof. (1) The set $E(S * X)$ is in bijective correspondence with the set X via the map $(p(x), x) \mapsto x$. By definition $s \cdot (p(x), x) = (t, y)(t, y)^*$ where $(t, y)(p(x), x) = (t, y)$ and $\pi_X(t, y) = sp(x)$. It follows that $t = sp(x)$. Now $(t, y)(t, y)^* = (tt^{-1}, t \cdot y)$ and $y = (t) \cdot x$. It follows that $(t, y)(t, y)^* = (p(s \cdot x), s \cdot x)$. Hence the two étale actions are naturally isomorphic.

(2) Given an étale map $\theta: T \rightarrow S$, we may construct an étale action $S \times E(T) \rightarrow E(T)$. Hence we may construct an étale map $\pi_{E(T)}: S * E(T) \rightarrow S$. Define $\alpha: T \rightarrow S * E(T)$ by $\alpha(t) = (\theta(t), t^*t)$. This is well-defined and $\pi_{E(T)}\alpha = \theta$. Observe that α is a bijection by Proposition 2.3. It remains to show that it is a $*$ -homomorphism which is routine. \square

We have therefore proved Theorem 1.1.

We now consider the case of étale actions *with global support*. Let (S, X, p) be such an action. Then $\pi_X: S * X \rightarrow S$ is a surjective étale map from the right generalized inverse $*$ -semigroup by Proposition 2.5. But by Proposition 2.4, the kernel of ϕ_X is γ . It follows that the $*$ -semigroup $S * X$ contains all the essential information about the action (S, X, p) .

Theorem 2.8. *The category of étale actions of inverse semigroups with global support is equivalent to the category of right generalized inverse $*$ -semigroups.*

Proof. We first define two functors.

Let $\theta: S \rightarrow T$ be a homomorphism between right generalized inverse $*$ -semigroups. Observe that $s\gamma t$ implies that $\theta(s)\gamma\theta(t)$. We may therefore define a homomorphism $\theta_1: S/\gamma \rightarrow T/\gamma$. There is also a homomorphism $\theta_2: E(S) \rightarrow E(T)$. Let

$(S/\gamma, E(S), p_S)$ and $(T/\gamma, E(T), p_T)$ be étale actions with global support associated with S and T respectively. The fact that (θ_1, θ_2) is a morphism of étale sets follows readily from the definition of the actions and the fact that θ is a $*$ -homomorphism.

Now let $(\alpha, \beta): (S, X, p) \rightarrow (T, Y, q)$ be a morphism of étale actions with global support. Define $\theta: S * X \rightarrow T * Y$ by $\theta(s, x) = (\alpha(s), \beta(x))$. It is routine to check that this is a well-defined map and a $*$ -homomorphism.

The fact that these two functors yield an equivalence of categories essentially follows by Proposition 2.7. \square

Let T be a generalized inverse semigroup. We say that T is *over the inverse semigroup* T/γ . The proof of the following is now immediate.

Corollary 2.9. *Let S be an inverse semigroup. Then the category of étale actions of S with global support is equivalent to the category of right generalized inverse $*$ -semigroups over S .*

We now describe an example that provided intuition for our construction. Let H be a group that acts on a set X on the left. Then $G = H \times X$ can be endowed with the structure of a groupoid in a construction that goes back to Ehresmann. The question arises of how we might capture the action purely algebraically. We can regard X as a right zero semigroup and then G becomes a right group: that is, a direct product of a group and a right zero semigroup. This leads to a forgetful functor from left actions of G to right groups with base group G . If $(h, x) \in G$ then its domain is x via the identification of (x, id) with x , and its range is hx . However, when we pass from the groupoid G to the right group G we lose information about ranges. A way to record this information in G is to preserve the inversion of the groupoid by considering G as a $*$ -semigroup by defining $(h, x)^* = (h, x)^{-1} = (h^{-1}, hx)$. Then the range of (h, x) is recorded as the domain of $(x, h)^*$. The axioms of $*$ -semigroup given in [3] are enough to prove that left actions of H are the same thing as right groups with base groups G that have the structure of $*$ -semigroups. In our theorem above, we are essentially replacing the group by an inductive groupoid.

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